

M-THEORY ON SEVEN MANIFOLDS WITH G -FLUXES

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Abstract

We compactify M-theory on seven-manifolds with a warp-factor and G -fluxes on the internal space. Because of non-zero G -fluxes, we are forced to adopt a Majorana supersymmetry spinor ansatz which does *not* have the usual direct product structure of two lower dimensional Majorana spinors. For the spinor ansatz that we choose, we find that supersymmetry puts strong constraints on the internal space namely that it must be conformal to a Ricci-flat seven-manifold of the form $X^7 = X^6 \times X^1$. The holonomy of X^6 must be larger than 11 if the warp-factor is to be non-trivial. The warp-factor depends only on the X^1 direction and is singular. We argue that to avoid this singularity one has to embed this solution in a Hořava-Witten setup and thus has natural links to much studied brane-world scenarios.

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The purpose of this paper is to examine the role of warp-factors and G -fluxes in supersymmetric compactifications of M-theory on Ricci-flat 7-manifolds so that the resulting theory has 4- d Minkowski slices ($\mathbb{R}^{1,3}$). M-theory, which is yet to be fully understood, admits as it's low energy limit 11- d supergravity [1]. In four-dimensions spinors can not be chiral and Majorana simultaneously. This subtlety becomes relevant when compactifying supergravity theories down to four dimensions with warp-factors that preserve some fraction of supersymmetry. In some recent works [9, 11, 10] on warped compactifications of M-theory on seven-manifolds this seems to have been overlooked.

For our spinor ansatz we adopt the one proposed in [12] and find as a consequence that our internal manifold has to be conformal to a Ricci-flat 7-manifold of the form $X^7 = X^6 \times X^1$. Therefore our internal space is not a *bona fide* 7-manifold like a G_2 -holonomy space, which has received a lot of attention recently, but our calculation points to certain pitfalls one has to circumvent if one is to compactify on 7-manifolds.

A few years ago it was shown by Becker and Becker [6] that when we turn on certain components of the background G -fluxes in M-theory compactified on a 8-manifold, supersymmetry requires that internal space to be a Calabi-Yau 4-fold. The external space of this solution is $\mathbb{R}^{1,2}$. This work suggests that it might be possible to do the something similar with $\mathbb{R}^{1,3}$ and an internal Ricci-flat manifold.

Some historical facts about compactification on 7-manifolds

An older review of the subject is [5]. Perhaps, the most famous solution is the one by Freund and Rubin [2]. By choosing an ansatz such that G is proportional to the volume-form of the four-dimensional Lorentzian subspace, one obtains a solution of the form $AdS^4 \times X^7$, where X^7 is a 7- d positive curvature Einstein space. For maximal supersymmetry X^7 is usually taken to be S^7 . This has been dubbed “spontaneous compactification”. It is not possible to flatten the AdS^4 without decompactifying the internal S^7 . This is not the only problem besetting spontaneous compactification. Another problem is the absence of chiral multiplets. This shortcoming was rectified in the heterotic version of M-theory by Hořava and Witten [21]. It has been realised recently that compactifying on G_2 -holonomy spaces with some special singularities can also be used to generate chiral multiplets [27]. But for the question we are interested in the most interesting solution is that of [12]. The solution presented in [12] is a warped product of AdS^4 and S^7 . Their ansatz for the G -fluxes is like that of Englert [4]. The Englert ansatz consists of, in addition to a Freund-Rubin piece, turning on those components of G which live exclusively on the internal manifold. These are the only components that can be turned on if one is to have a maximally symmetric space-time as the external space. The Englert solution is not in general supersymmetric. In [12] the G -fluxes are related to the torsion on S^7 via the Maxwell equation and the resulting theory has G_2 -invariance and thus $N = 1$ supersymmetry. The claim is that this solution corresponds to the G_2 -invariant point of the superpotential of the 4- d $N = 8$ gauged supergravity theory of de Wit and Nicolai [13]. The important point of their solution is that their 11- d Killing spinor, η , is not a direct product of 4- and 7-dimensional Majorana spinors. This subtlety does not arise in the simplest S^7 compactifications and compactifications on manifolds with dimension other than seven¹. While the usual spinor

¹ See for example [6].

ansatz $\eta = \epsilon \otimes \theta$, with ϵ and θ are 4- d anticommuting and 7- d commuting Majorana Killing spinors, simplify life considerably, it restricts the space of supersymmetric solutions to those without warp-factors.

It has been argued in [14, 15] that the solutions put forward in [6, 26] do not exist when one considers the compactification on *compact* manifolds. However, it was also noted in [15] that this type of solutions can exist when the M5-brane anomaly term is added to the 11- d supergravity action [16].

Outline

We now present an outline of the sections of our paper. The next section of this paper is to fix our conventions. In section 3 we present our main calculations related to supersymmetry and obtain constraints on the compactification space and the warp-factor. In section 4 we give a geometric interpretation of the supersymmetry constraints and show that the internal space is conformal to $X^6 \times X^1$. The reducibility of the internal space is seen as a consequence of the Killing spinor ansatz that we have chosen. What emerges from this discussion is that the warp-factor can have non-trivial dependence only along X^1 . We then solve Einstein's equation to find the shape of the warp-factor which unfortunately turns out to be singular. We then close with some comments about what could be the application of such a solution. In the appendix we discuss the four-seven split of 11- d Majorana spinor and give some Clifford algebra identities used in section 3.

2 Metric Ansatz and Conventions

The bosonic part of the action for 11- d supergravity is [1]

$$\mathcal{S}_{11} = \frac{1}{2} \int d^{11}x \sqrt{-\bar{g}} \bar{R} - \int \left[\frac{1}{4} \bar{G} \wedge \bar{*} \bar{G} + \frac{1}{12} \bar{C} \wedge \bar{G} \wedge \bar{G} \right], \quad (2.1)$$

where the first term is the Einstein-Hilbert term and \bar{C} is the 3-form potential and $\bar{G} = d\bar{C}$ is the corresponding 4-form field strength. The only fermionic matter field in this theory is the gravitino, ψ_M , which is a spinor-vector, whose variation under supersymmetry is given by

$$\delta_\eta \psi_M = \bar{\nabla}_M \eta - \bar{Z}_M \eta, \quad (2.2)$$

where η is the Majorana supersymmetry parameter and

$$\bar{Z}_M = \frac{1}{288} \left[\bar{\Gamma}_M^{PQRS} - 8\delta_M^P \bar{\Gamma}^{QRS} \right] \bar{G}_{PQRS} \quad (2.3)$$

$$\bar{\Gamma}^{M_1 M_2 \dots M_n} = \bar{\Gamma}^{[M_1} \bar{\Gamma}^{M_2} \dots \bar{\Gamma}^{M_n]}. \quad (2.4)$$

If a bosonic configuration is to preserve supersymmetry then η must satisfy the Killing spinor equation,

$$\bar{\nabla}_M \eta - \bar{Z}_M \eta = 0. \quad (2.5)$$

We adopt the usual warp-factor metric ansatz appropriate for $X_4 \times X_7$,

$$\bar{g}_{MN}(x, y) = D^{-1}(y) \begin{bmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{bmatrix} = D^{-1}(y) g_{MN}(x, y) \quad (2.6)$$

where we follow the convention that x^μ (y^m) denote the coordinates of the 4-dimensional (7-dimensional) space whose geometry is described by the metric $g_{\mu\nu}$ (g_{mn}).

We want to write out the Killing spinor equation in terms of the unbarred metrics and so we observe

$$\bar{\nabla}_M = \nabla_M - \frac{1}{4} D^{-1} \Gamma_M{}^N (\nabla_N D) \quad (2.7)$$

$$\bar{\Gamma}_M = D^{-\frac{1}{2}} \Gamma_M \quad \bar{\Gamma}^M = D^{\frac{1}{2}} \Gamma^M. \quad (2.8)$$

We shall assume that \bar{G}_{MNPQ} has zero conformal weight. Discussions on Clifford algebra identities and Majorana spinors relevant for the calculations are given in the appendix.

3 The $\mathbb{R}^{1,3}$ Solution with G -fluxes

The most general choice of G consistent with compactifications down to maximally symmetric 4- d space-time is given by the Englert ansatz [4]:

$$\begin{aligned} G_{\mu\nu\rho\sigma} &= 6m\epsilon_{\mu\nu\rho\sigma} \\ G_{mnpq} &\neq 0. \end{aligned} \quad (3.1)$$

For our solution we shall ultimately set the Freund-Rubin parameter m to zero. With the appropriate choice for the Dirac matrices (see appendix A for details) the four-dimensional part of (2.5) now becomes, expressed in terms of the unbarred metric,

$$\nabla_\mu \eta - \frac{\partial_m(\log D)}{4} (\gamma_\mu \Sigma^5 \otimes \gamma^m) \eta - \frac{D^{\frac{3}{2}}}{288} (\gamma_\mu \otimes \not{G}) \eta - im D^{\frac{3}{2}} (\gamma_\mu \Sigma^5 \otimes \mathbb{1}) \eta = 0 \quad (3.2)$$

where $\not{G} \equiv G_{mnpq} \gamma^{mnpq}$ and $\Sigma^5 \equiv \frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$ is the 4- d chirality operator. Now if we assume $\nabla_\mu \eta = 0$ for Minkowski space then we are left with the condition

$$\frac{\partial_m(\log D)}{4} (\Sigma^5 \otimes \gamma^m) \eta + \frac{D^{\frac{3}{2}}}{288} (\mathbb{1} \otimes \not{G}) \eta + im D^{\frac{3}{2}} (\Sigma^5 \otimes \mathbb{1}) \eta = 0 \quad (3.3)$$

The integrability for (3.2) for $\mathbb{R}^{1,3}$ yields the following condition

$$\begin{aligned} &\left[\frac{1}{16} \partial_m(\log D) \partial^m(\log D) - D^3 m^2 \right] (\mathbb{1} \otimes \mathbb{1}) \eta \\ &- \frac{D^3}{(288)^2} (\mathbb{1} \otimes \not{G}^2) \eta + \frac{D^{\frac{3}{2}} \partial_m(\log D)}{144} (\Sigma^5 \otimes G^m) \eta \\ &+ \frac{im D^{\frac{3}{2}} \partial_m(\log D)}{2} (\mathbb{1} \otimes \gamma^m) \eta = 0 \end{aligned} \quad (3.4)$$

where $G^m \equiv \gamma_{pqr} G^{mpqr}$. In the absence of a warp-factor this reduces to

$$\frac{1}{(288)^2} (\mathbb{1} \otimes \not{G}^2) \eta = -m^2 (\mathbb{1} \otimes \mathbb{1}) \eta. \quad (3.5)$$

This equation implies that η is an eigenspinor of \not{G}^2 with negative eigenvalue. Since \not{G} is a hermitian operator the eigenvalues of \not{G}^2 are positive semi-definite. So the above equation implies

$$\begin{aligned} m &= 0 \\ \not{G} \eta &= 0. \end{aligned} \quad (3.6)$$

This implies, via the internal-space Killing spinor equation integrability condition, that all components of G has to vanish together with the fact that the internal space has to be Ricci-flat [7].

When there is a non-constant warp-factor present and $m = 0$ (3.4) becomes

$$\frac{\partial_m(\log D)\partial^m(\log D)}{16}(\mathbb{1} \otimes \mathbb{1})\eta - \frac{D^3}{(288)^2}(\mathbb{1} \otimes \not{G}^2)\eta + \frac{D^{\frac{3}{2}}\partial_m(\log D)}{144}(\Sigma^5 \otimes G^m)\eta = 0 \quad (3.7)$$

This equation is satisfied if η is a simultaneous eigenspinor of $\frac{D^3 \not{G}^2}{(288)^2}$ and $\frac{D^{\frac{3}{2}}\partial_m(\log D)}{144}(\Sigma^5 \otimes G^m)$ with eigenvalues $\frac{\partial_m(\log D)\partial^m(\log D)}{16}$ and zero, respectively. These requirements are satisfied if

$$-\frac{\partial_n(\log D)}{24}(\mathbb{1} \otimes \gamma_m{}^n)\eta = \frac{D^{\frac{3}{2}}}{288}(\Sigma^5 \otimes G_m)\eta. \quad (3.8)$$

It is easy to see that this condition implies the one suggested by (3.3) with $m = 0$,

$$-\frac{\partial_n(\log D)}{4}(\Sigma^5 \otimes \gamma^n)\eta = \frac{D^{\frac{3}{2}}}{288}(\mathbb{1} \otimes \not{G})\eta. \quad (3.9)$$

The Clifford algebra identities given in the appendix are useful in these manipulations.

Structure of the Killing Spinor

What does (3.9) tells us about the structure of the Killing spinor? It was noted in [12] that turning on G -fluxes forces us to consider Killing spinors which can not be of the form $\eta = \epsilon \otimes \theta$, where ϵ and θ are Majorana and pseudo-Majorana Killing spinors on $\mathbb{R}^{1,3}$ and the internal space X^7 , respectively. We can see that very easily from the foregoing conditions. Let us suppose, for the sake of argument, that the Majorana Killing spinor has the form

$$\eta = \epsilon \otimes \theta. \quad (3.10)$$

We assume θ to be pseudo-Majorana and commuting. Since η is Majorana and anticommuting, ϵ must be Majorana and anticommuting. Then suppose that (3.9) is satisfied by

$$-\frac{\partial_n(\log D)}{4}\gamma^n\theta = \frac{D^{\frac{3}{2}}}{288}\not{G}\theta = \theta' \quad (3.11)$$

for some θ' . We now have two cases. If $\theta' \neq 0$ then we arrive at the contradiction

$$\Sigma^5\epsilon = \epsilon. \quad (3.12)$$

If on the other hand $\theta' = 0$ then by squaring the left hand side of (3.11) we obtain

$$\partial_m(\log D)\partial^m(\log D) = 0 \quad (3.13)$$

which means that the warp-factor is constant and we are left with a special case of (3.6) and hence $G = 0$. This complication doesn't occur, for example, in compactification on 8-manifolds [6] where one can define the internal spinors to be chiral and Majorana

simultaneously. So the only way to introduce non-trivial G -flux and warp-factor is to introduce a non-minimal structure of η . Following [12] we make the ansatz

$$\eta = \left[A + B_m \left(\Sigma^5 \otimes \gamma^m \right) \right] (\epsilon \otimes \theta) \equiv \mathcal{A}(\epsilon \otimes \theta) \quad (3.14)$$

where A and B_m are assumed to be real functions of y^m . Reality is necessary for η to a Majorana spinor.

We insert our ansatz (3.14) into (3.8) and since ϵ and $\Sigma^5 \epsilon$ are linearly independent, we obtain

$$-\frac{D^{\frac{3}{2}}}{288} G^m B_p \gamma^p \theta = \frac{\partial_n (\log D) A}{24} \gamma^{mn} \theta \quad (3.15)$$

and

$$-\frac{D^{\frac{3}{2}} A}{288} G^m \theta = \frac{\partial_n (\log D)}{24} \left[B^p \gamma^{mn} \theta - B^m \gamma^n + B^n \gamma^m \right] \theta. \quad (3.16)$$

Contracting with γ_m we obtain

$$-\frac{A}{4} \partial_m (\log D) \gamma^m \theta = \frac{D^{\frac{3}{2}}}{288} [B_m \gamma^m \not{G} - 8 B_m G^m] \theta \quad (3.17)$$

and

$$\frac{\partial_n (\log D)}{4A} [B_m \gamma^{mn} - B^n] \theta = \frac{D^{\frac{3}{2}}}{288} \not{G} \theta. \quad (3.18)$$

These condition are of course the conditions that we get by putting our ansatz into (3.9).

Let us now see the effect of (3.8) on the internal space Killing spinor equation. This equation is

$$\nabla_m \eta - \frac{1}{4} \partial_n (\log D) (\mathbb{1} \otimes \gamma_m{}^n) \eta - \frac{D^{\frac{3}{2}}}{288} \left[\Sigma^5 \otimes \gamma_m{}^{pqrs} G_{pqrs} - 8 \Sigma^5 \otimes G_m \right] \eta = 0. \quad (3.19)$$

By using

$$\gamma_m{}^{pqrs} G_{pqrs} = \gamma_m \not{G} - 4 G_m$$

we get

$$\nabla_m \eta - \frac{\partial_n (\log D)}{4} (\mathbb{1} \otimes \gamma_m{}^n) \eta - \frac{D^{\frac{3}{2}}}{288} \left[\Sigma^5 \otimes \gamma_m \not{G} - 12 \Sigma^5 \otimes G_m \right] \eta = 0. \quad (3.20)$$

By using (3.8) we obtain

$$\nabla_m \eta + \frac{1}{4} \partial_m (\log D) (\mathbb{1} \otimes \mathbb{1}) \eta - \frac{1}{2} \partial_n (\log D) (\mathbb{1} \otimes \gamma_m{}^n) \eta = 0. \quad (3.21)$$

Now defining

$$\tilde{\eta} = D^{\frac{1}{4}} \eta \quad (3.22)$$

$$\hat{g}_{mn} = D^{-2} g_{mn} \quad (3.23)$$

we obtain

$$\hat{\nabla}_m \tilde{\eta} = 0. \quad (3.24)$$

This equation ensures that the internal space is conformal to a Ricci-flat manifold with holonomy contained within G_2 . To obtain a nontrivial solution we demand that $\tilde{\theta} = D^{\frac{1}{4}}\theta$ be covariantly constant with respect to \hat{g}_{mn} . So we get

$$(\hat{\nabla}_m \mathcal{A})(\epsilon \otimes \tilde{\theta}) = 0. \quad (3.25)$$

The solution to (3.25) consistent with the conditions (3.17-3.18) is

$$A = \text{Constant}. \quad (3.26)$$

and

$$B_m = \pm \frac{\partial_m(\log D)}{\sqrt{\partial_n(\log D)\partial^n(\log D)}}. \quad (3.27)$$

where $\hat{\nabla}\hat{\mathbf{B}} = 0$.

4 Geometric Interpretation

If \mathcal{X} is a holonomy- \mathcal{H} Riemannian manifold and there exists on \mathcal{X} a tensor field \mathbf{T} such that

$$\nabla \mathbf{T} = 0 \quad (4.28)$$

then it is a fundamental result of the theory of holonomy groups [23] that \mathbf{T} must be invariant under the \mathcal{H} . I.e.

$$\Lambda \mathbf{T} = \mathbf{T} \quad \forall \Lambda \in \mathcal{H}. \quad (4.29)$$

Let us apply this result to our case. Let us denote by X^7 the Ricci-flat manifold described by the metric defined in (3.23), which dropping the hat, we denote from now on as g_{mn} . In this section we work with this metric and we drop all hats from all formulae. Equation (3.24) implies that it's holonomy group, $\mathcal{H} \subset G_2$. Since \mathbf{B} is a vector field then we can write it locally as

$$\mathbf{B} = \frac{\partial}{\partial z}. \quad (4.30)$$

If the tangent space splits up under the action of the holonomy group locally then this implies via a fundamental theorem in differential geometry [24] that $X^7 = X^6 \times X^1$. It also means that \mathbf{B} can only be a function of the coordinate of X^1 which we denote by z . Therefore of all the possibilities we must exclude the ones with $\mathcal{H} = G_2$.

One way of understanding the exclusion of G_2 solutions is as follows. G_2 -holonomy spaces admit only one commuting pseudo-Majorana spinor, $\tilde{\theta}$ [17, 19, 20]. However since $\tilde{\theta}\gamma^m\tilde{\theta} = 0$, there is no covariantly constant vector associated to this spinor. And we have seen that for our ansatz supersymmetry demands such a covariantly constant vector. It seems to us that to obtain a G_2 -holonomy solution one should adopt a spinor ansatz of the form

$$\eta = \left[A + B_{pqr} \left(\Sigma^5 \otimes \gamma^{pqr} \right) \right] (\epsilon \otimes \theta). \quad (4.31)$$

We wish to investigate this ansatz further in a future communication.

Since \mathbf{B} is unit norm (4.28) does not put any more constraints on the warp-factor. So to determine the shape of the warp-factor we have to solve the bosonic equations which we do in the next section. Our task has been simplified since the warp-factor can depend only on z .

$$\phi_{mnp} = i \bar{\tilde{\theta}} \gamma_{mnp} \tilde{\theta} \quad (4.32)$$

and so its Hodge-dual with respect to g_{mn} is given by

$$*\phi_{mnpq} = \bar{\tilde{\theta}} \gamma_{mnpq} \tilde{\theta}. \quad (4.33)$$

From (3.16) we see

$$\phi_{pqr} G^{mpqr} = 0. \quad (4.34)$$

This equation gives us the non-zero components of G depending on the choice of the seven-manifold. Equation (3.18) then encodes the relationship between the dual 4-form, $*\phi$, and the non-zero components of G . We now do a case by case study of the various possibilities for X^7 classified by their holonomy groups \mathcal{H} . In the following subsections we work exclusively with tangent space indices which we denote by a prime (') on the index. For example, an orthonormal basis will be denote by $\{e_{a'}\}$. The constructions that follow are very similar to what was done without a warp-factor in [18].

4.1 $\mathcal{H} = SU(3)$

We set $X^6 = CY_3$. On a CY_3 ϕ is given by

$$\phi = \omega \wedge e_{7'} + \text{Im}(\Omega) \quad (4.35)$$

where ω and Ω are the Kähler 2-form and the $SU(3)$ -structure 3-form, respectively. They are given by

$$\omega = e_{1'} \wedge e_{2'} + e_{3'} \wedge e_{4'} + e_{5'} \wedge e_{6'}$$

$$\Omega = (e_{1'} + ie_{2'}) \wedge (e_{3'} + ie_{4'}) \wedge (e_{5'} + ie_{6'}).$$

We can see from (4.34) that only seven components of G can be non-zero.

4.2 $\mathcal{H} = SU(2)$

We can take $X^6 = K3 \times T^2$. Then ϕ is given by

$$\begin{aligned} \phi &= \omega_{1'} \wedge e_{5'} + \omega_{2'} \wedge e_{6'} + \omega_{3'} \wedge e_{7'} + e_{5'} \wedge e_{6'} \wedge e_{7'} \\ &+ \text{cyclic perm. of } e_{5'}, e_{6'} \text{ and } e_{7'} \text{ in the first three terms.} \end{aligned} \quad (4.36)$$

where $\omega_{1'}$ is the Kähler 2-form and $\omega_{2'} + i\omega_{3'}$ is the complex volume 2-form of $K3$. They are given by

$$\begin{aligned} \omega_{1'} &= e_{1'} \wedge e_{4'} + e_{2'} \wedge e_{3'} \\ \omega_{2'} &= e_{1'} \wedge e_{3'} - e_{2'} \wedge e_{4'} \\ \omega_{3'} &= e_{1'} \wedge e_{2'} + e_{3'} \wedge e_{4'}. \end{aligned}$$

We can now see from (4.34) that only one component of G can be non-zero and it the one proportional to the volume form of $K3$.

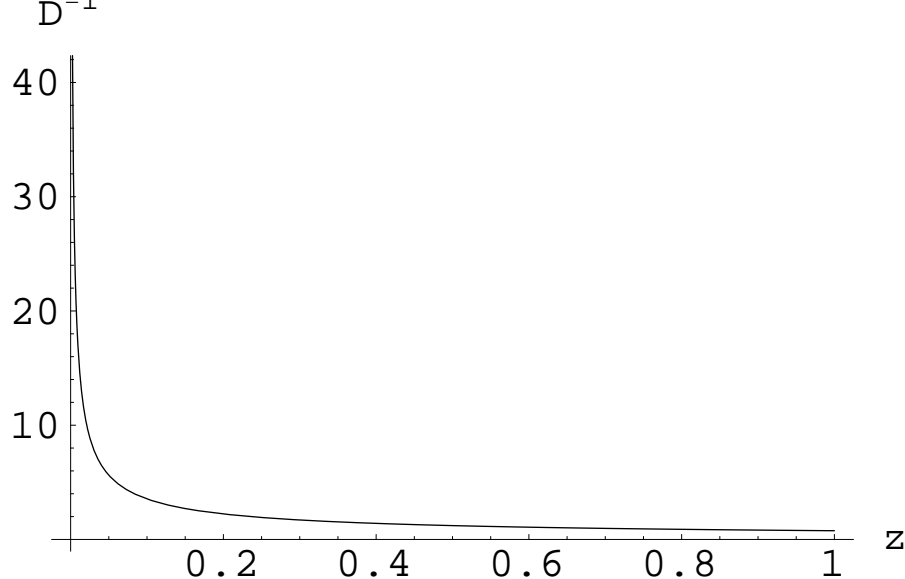


Figure 1: Shape of the warp-factor.

Shape of the Warp-Factor

If we have assumed that there are no M2- or M5-brane sources then supersymmetry restricts us to look for solutions of the form

$$d\bar{s}^2 = D^{-1}(z) ds_{\mathbb{R}^{1,3}}^2 + D(z) \{dz^2 + ds_{X^6}^2\}. \quad (4.37)$$

Einstein's equation then reduces to

$$\frac{3}{2} \left(\frac{d \log D}{dz} \right)^2 = - \frac{d^2 \log D}{dz^2} \quad (4.38)$$

Eq.(4.38) is solved easily to give

$$D(z) = e^{a_2} \left| a_1 - \frac{3}{2} z \right|^{\frac{2}{3}} \quad (4.39)$$

where a_1 and a_2 are integration constants. The metric is singular at $z = \frac{2a_1}{3}$. The singularity, however, is repulsive for massive particles.

5 Discussion

So we have seen that to have warped compactification of M-theory down to four-dimensional Minkowski space with G -fluxes on the internal space we need to consider more complicated spinor ansatz than usually assumed. The fact that our solution is singular is not surprising in the context of the no-go theorem of [14, 15]. However, the reducibility of the internal space in our case is due to the fact that our spinor ansatz contains a covariantly constant vector field. An ansatz with a covariantly constant 3-form field should then be the appropriate ansatz for G_2 -holonomy compactification. For such compactifications one should

allow for singular internal spaces (and thus possibly side step the no-go theorem) or the type explored in [27] to generate chiral fermions.

A possible application of the simple model discussed here could be in the context of Hořava-Witten theory. There 11- d supergravity lives in the bulk and one of the direction is a line which terminates on 10- d “branes”. An observer living in the bulk sees usual 11- d supergravity but the physics on the branes are quite different, containing chiral multiplets transforming under E_8 or \bar{E}_8 . So it should be possible to embed our solution in the bulk of Hořava-Witten theory and put one the branes before one can reach the singularity. In our solution 6 of the dimensions have been compactified and so in effect we have a theory where there are two 4- d Minkowski branes living on the ends of a line. Since the metric is warped it is clear that observers on each of these branes will observe different physics related to the “fifth” dimension. This is a supersymmetric analog of the much studied brane-world scenarios [25].

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A Dirac Matrix Identities

We adopt the convention in which 11- d space-time has signature $\{-++\dots+\}$. The time-like Dirac matrix is anti-hermitian and the rest are hermitian. They obey

$$\{\Gamma_M, \Gamma_N\} = 2g_{MN}\mathbb{1}. \quad (\text{A.1})$$

The Ricci identity for spinors is

$$[\nabla_M, \nabla_N]\psi = \frac{1}{4}R_{MNPQ}\Gamma^{PQ}\psi \quad (\text{A.2})$$

and spheres in our convention have positive curvature. Decomposition of the Dirac matrices appropriate for four-seven split is

$$\begin{aligned} \Gamma_\mu &= \gamma_\mu \otimes \mathbb{1} \\ \Gamma_m &= \Sigma^5 \otimes \gamma_m \end{aligned} \quad (\text{A.3})$$

where we have defined

$$\Sigma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma. \quad (\text{A.4})$$

$\epsilon_{\mu\nu\rho\sigma}$ is the 4- d Levi-Civita tensor density. In the text we also make use of

$$\gamma^{\nu\rho\sigma} = i\Sigma^5\gamma_\mu\epsilon^{\nu\rho\sigma\mu} \quad (\text{A.5})$$

and so

$$\epsilon_{\mu\nu\rho\sigma}\gamma^{\nu\rho\sigma} = 6i\Sigma^5\gamma_\mu \quad (\text{A.6})$$

Let us now turn to an important issue about Majorana spinors. On 7- d Euclidean manifolds, it only possible to define *pseudo*-Majorana spinors which means that the charge conjugation matrix is symmetric [8]. The transpose of the Dirac matrices are given by

$$\gamma_m^T = -C\gamma_m C^{-1} \quad (\text{A.7})$$

and the pseudo-Majorana condition is then given by

$$\bar{\theta} = \theta^T C. \quad (\text{A.8})$$

If θ is a *commuting* pseudo-Majorana spinor then we easily obtain

$$\begin{aligned} \bar{\theta} \gamma^{m_1 \dots m_n} \theta &= \theta^T C \gamma^{m_1 \dots m_n} \theta \\ &= (-1)^{\frac{n^2+n}{2}} \bar{\theta} \gamma^{m_1 \dots m_n} \theta \end{aligned} \quad (\text{A.9})$$

so we see that the antisymmetric tensor $\bar{\theta} \gamma^{m_1 \dots m_n} \theta$ vanishes except for $n = 0, 3, 4, 7$. In a similar manner it can be shown that the analogous tensor in 11- d Lorentzian signature space made out of an *anticommuting* Majorana spinor η vanishes except for $n = 0, 3, 4, 7, 8, 11$.

A useful identity that lets us expand the product of two elements of the Clifford algebra as a linear combination is

$$\begin{aligned} \Gamma_{M_1 \dots M_m} \Gamma^{N_1 \dots N_n} &= \Gamma_{M_1 \dots M_m}^{N_1 \dots N_n} + mn \Gamma_{M_1 \dots M_{m-1}}^{N_2 \dots N_n} \delta_{M_m}^{N_1} \\ &+ \frac{1}{2!} m(m-1)n(n-1) \Gamma_{M_1 \dots M_{m-2}}^{N_3 \dots N_n} \delta_{M_{m-1}}^{N_1} \delta_{M_m}^{N_2} + \dots \end{aligned} \quad (\text{A.10})$$

from which one can derive the following identities for the seven-dimensional Dirac matrices,

$$\begin{aligned} [\gamma_m, \gamma^n] &= 2\gamma_m{}^n & \{\gamma_m, \gamma^n\} &= 2\delta_m{}^n \\ \{\gamma_{mn}, \gamma^r\} &= 2\gamma_{mn}{}^r & [\gamma_{mn}, \gamma^r] &= -4\delta^r{}_{[m} \gamma_{n]} \\ [\gamma_{mnp}, \gamma^r] &= 2\gamma_{mnp}{}^r & \{\gamma_{mnp}, \gamma^r\} &= 6\delta^r{}_{[m} \gamma_{np]} \\ \{\gamma_{mnpq}, \gamma^r\} &= 2\gamma_{mnpq}{}^r & [\gamma_{mnpq}, \gamma^r] &= -8\delta^r{}_{[m} \gamma_{npq]} \\ [\gamma_{mnpqk}, \gamma^r] &= 2\gamma_{mnpqk}{}^r & \{\gamma_{mnpqk}, \gamma^r\} &= 10\delta^r{}_{[m} \gamma_{npqk]}. \end{aligned} \quad (\text{A.11})$$

In our convention $\mathbb{1}, \gamma^m, \gamma^{mnpq}, \gamma^{mnpqr}$ are hermitian while the rest of the elements of the Clifford algebra on the internal space are antihermitian.

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